Problem 6.7.1. A binomial random variable $X(n, p)$ that counts the number of heads in n independent tosses of a biased coin (that has probability p of hitting heads and probability $(1-p)$ of hitting tails). Argue that

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

for all $k \in \{0, 1, \ldots, n\}$. Find the expected value of X.

Problem 6.7.2. At a post-Covid Yale party, n guys who love hats start the following strange experiment: each of them puts his hat in the center of a room, where the hats are all mixed together at random. These are all the same hat they both from the Yale Bookstore so they are indistinguishable from each other once mixed, except for the fact everyone loves his own hat that much that they have their name knitted somewhere on the inside. Each person takes turns and selects a hat from the mixed pile and leaves (with the hat) if they get their own hats (after checking the names inside). If not, they put the hats back and after everyone is done they mix them again. This goes on until everyone gets his own hat.

(a) What's the expected number of people who manage choose their own hat in round 1? What Example problem above is this question related to?

(b) Find the expected number of rounds needed for the experiment until everyone gets his own hat.

Problem 6.7.3. Let X be a random variable taking integral nonnegative values. Prove that

$$P(X = 0) \leq \frac{\mathrm{Var}(X)}{E(X^2)}.$$

Problem 6.7.4. Let $x_{ij} \in [0, 1]$ for $1 \leq i, j \leq n$. Prove that

$$\prod_{j=1}^{n}\left(1 - \prod_{i=1}^{m} x_{ij}\right) + \prod_{i=1}^{m}\left(1 - \prod_{j=1}^{n}(1 - x_{ij})\right) \geq 1.$$

Problem 6.7.5. Prove that there exists a 2-coloring of K_n with at most $\binom{n}{a} 2^{1-\binom{a}{2}}$ monochromatic K_a's.

Problem 6.7.6. Let $A_1, A_2, \ldots A_k$ be subsets of $\{1, 2, \ldots, n\}$ with $|A_i| = 3$ for all $i = 1, \ldots, k$. Show that it is possible to color the elements of $\{1, 2, \ldots, n\}$ with c colors in such a way that at most k/c^2 of the A_i are monochromatic.

Problem 6.7.7. Prove that there is an absolute constant $c > 0$ with the following property. Let A be an $n \times n$ matrix with pairwise distinct entries. Then there is a permutation of the rows of A so that no column in the permuted matrix contains an increasing subsequence of length at least $c\sqrt{n}$.

Problem 6.7.8. Let V be a set with n elements and let E be a family of m subsets of V, each having three elements. Prove that if $3m \geq n$, then there exists $S \subset V$ having at last $\frac{2n}{3}\sqrt{\frac{n}{3m}}$ elements and such that no element of E is contained in S.

Problem 6.7.9. Let F be a family of subsets of $[n]$ such that we cannot find $A_1, ..., A_{s+1} \in F$ pairwise distinct and for which $A_1 \subset A_2 \subset ... \subset A_{s+1}$. Then

$$\sum_{A \in F} \frac{1}{\binom{n}{|A|}} \leq s.$$

Problem 6.7.10. (Bollobas) Let $A_1, A_2, ..., A_n$ and $B_1, B_2, ..., B_n$ be distinct sets of positive integers such that $A_i \cap B_i = \emptyset$ for all i, but $A_i \cap B_j \neq \emptyset$ for all $i \neq j$. Then

$$\sum_{i=1}^{n} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1.$$

Problem 6.7.11. (Tusza) Let $A_1, A_2, ..., A_n$ and $B_1, B_2, ..., B_n$ be distinct subsets of $\mathbb{N}$ such that $A_i \cap B_i = \emptyset$ for all i and $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset$ for all $i \neq j$. Prove that for all $p \in [0, 1]$

$$\sum_{i=1}^{n} p^{|A_i|}(1-p)^{|B_i|} \leq 1.$$

Problem 6.7.12. (Alon) The minimal degree of the vertices of a graph G with n vertices is $d > 1$. Prove that there exists a subset S of the vertices such that

1) $|S| \leq n \cdot \frac{1+\ln(d+1)}{d+1}$.

2) Any vertex of G is either in S or neighbor of a vertex in S.

The last problem is a difficult problem which is beyond the scope of the material, but I wanted to include it for you to see how Chebyshev's inequality gets used in practice (for problems that do not explicitly look like they might need it).

Problem 6.7.13. Let $v_1, v_2, ..., v_n$ be n vectors in the plane, whose coordinates are integers of absolute value less that $\frac{1}{100}\sqrt{\frac{2^n}{n}}$. Prove that there are disjoint subsets I, J of $\{1, 2, ..., n\}$ such that $\sum_{i \in I} v_i = \sum_{j \in J} v_j$.